Cauchy's theorem:

If f(z) is analytic function and $f'(z)$ is continuous at each point within and on a closed curve C then

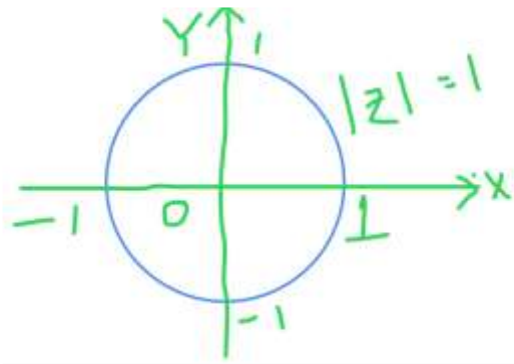
$$\int_C f(z)dz = 0$$

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$ if C is the circle $|z| = 1$

Let f(z) $= \frac{e^z}{z-2}$ and it is analytic inside and on the given circle $|z| = 1$. For the circle $|z| = 1$ we have z=2 outside the circle.

Hence by Cauchy's theorem

$\oint_C \frac{e^z}{z-2}dz = 0 \Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz = 0$



Cauchy's integral formula:

If f(z) is analytic function within and on a simple closed curve C in the domain R and z=a is any point within C then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}dz$ if C is the circle $|z - 1| = 4$

Let f(z) $= e^{3z}$ and it is analytic inside and on the given circle $|z-1| = 4$. Again $z = \pi i$ is a point inside the given circle. Then by Cauchy's integral formula, we have

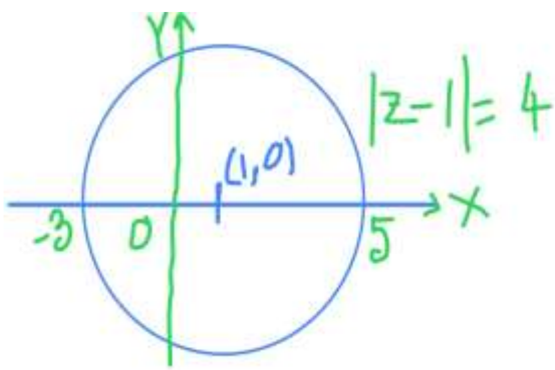
$\oint_C \frac{e^{3z}}{z-\pi i}dz = 2\pi i\, f(\pi i)$

$= 2\pi i\, e^{3\pi i}$

$= 2\pi i\, (cos\, 3\pi + isin3\pi)$

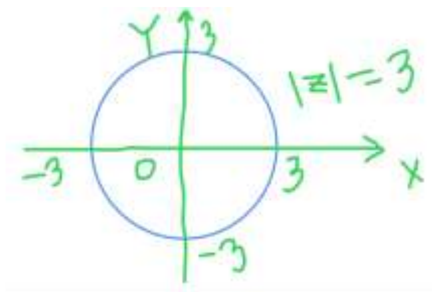
$= 2\pi i\, (-1 + 0)$

$= -2\pi i$



Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz$ if $t > 0$ and C is the circle $|z| = 3$

Let f(z) $= e^{zt}$ and it is analytic inside and on the given circle $|z| = 3$.



Now

$$\oint_C \frac{e^{zt}}{z^2+1}dz = \oint_C \frac{f(z)}{(z+i)(z-i)}dz$$

$= \frac{1}{2i}\oint_C \left(\frac{1}{z-i} - \frac{1}{z+i}\right) f(z)dz$

$= \frac{1}{2i}\oint_C \frac{f(z)}{z-i}dz - \frac{1}{2i}\oint_C \frac{f(z)}{z+i}dz$

Since both z=-i and z=i inside the given circle. Then by Cauchy's integral formula, we have

$\oint_C \frac{e^{zt}}{z^2+1}dz = \frac{1}{2i}\oint_C \frac{f(z)}{z-i}dz - \frac{1}{2i}\oint_C \frac{f(z)}{z+i}dz$

$= \frac{1}{2i}2\pi i\, f(i) - \frac{1}{2i}2\pi i\, f(-i)$

$= \pi\, f(i) - \pi\, f(-i)$

$= \pi e^{it} - \pi e^{-it}$

$= \pi\left(e^{it} - e^{-it}\right)$

$= \pi.\, 2i\, sint$

$\Rightarrow \oint_C \frac{e^{zt}}{z^2+1}dz = 2\pi isint$

$\Rightarrow \boldsymbol{\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz = sint}$

Self exercise:

- ❖ Evaluate $\oint_C \frac{dz}{z+5}dz$ if C is the circle $|z| = 2$
- ❖ Evaluate $\frac{1}{2\pi i}\oint_C \frac{z^2+1}{(z-1)(z-2)}dz$ if $t > 0$ and C is the circle $|z-1| = 2$
- ❖ Evaluate $\oint_C \frac{dz}{z+1}dz$ if C is the circle $|z| = 2$